Metric spaces and computability theory

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Joint works with Melnikov, and Gavruskin

K-trivial sets of natural numbers

- K-trivial sets were first studied by Chaitin and Solovay in 1975. They form a key class at the interface of computability and randomness.
- Coincidence results were obtained from the early 2000s (Hirschfeldt, Downey, N., Stephan, ...). K-trivial sets are
  - far from random by definition
  - weak as an oracle
  - computably approximable with a finite total cost of changes.
- K-trivials induce an ideal in the Δ^0_2 Turing degrees generated by its c.e. members.

We will extend the notion of K-triviality to the more general setting of points in a computable metric space $\mathcal{M}$.

Main results on K-trivial points (Melnikov and N., 2014)

- Existence and preservation:
  - If a computable metric space $\mathcal{M}$ is perfect then it contains a K-trivial non-computable point.
  - K-triviality is preserved under computable maps between metric spaces.
- We define K-triviality of a point $x \in \mathcal{M}$ via fast converging Cauchy sequences of special points.

This global condition is equivalent to a "local" condition saying that the information content of special points closer and closer to x grows very slowly.

Computable metric spaces

Definition

Let (M, d) be a complete metric space, and let $(\alpha_i)_{i\in\mathbb{N}}$ be a dense sequence in M.

- $\mathcal{M} = (M, d, (\alpha_i)_{i\in\mathbb{N}})$ is a computable metric space if $d(\alpha_i, \alpha_k)$ is a computable real uniformly in i, k. (Repetition, i.e. distance value 0 even for $i \neq k$, is allowed.)
- We call the elements of the sequence $(\alpha_i)_{i\in\mathbb{N}}$ the special points. We call such $(\alpha_i)_{i\in\mathbb{N}}$ a computable structure of the metric space. Not necessarily unique.
- We often identify α_i with $i \in \mathbb{N}$, e.g. when using the special points as oracle queries.

Examples of computable structures: in $\mathbb{R}$ the rationals; in $\mathcal{C}[0, 1]$ the polygonal functions with rational breakpoints.

Cauchy names for points

Definition

- A sequence $(p_s)_{s\in\mathbb{N}}$ of special points is called a Cauchy name if $d(p_s, p_{s+1}) \leq 2^{-s-1}$ for each $s \in \mathbb{N}$.
- Since $\mathcal{M}$ is complete, $x = \lim_s p_s$ exists. We say that $(p_s)_{s\in\mathbb{N}}$ is a Cauchy name for x. Note that $d(x, p_s) \leq 2^{-s}$.
- A point x is computable if it has a computable Cauchy name.

Defining K: Prefix-free machines

A partial computable function from binary strings to binary strings is called prefix-free machine if its domain is an anti-chain under the prefix relation of strings.

There is a universal prefix-free machine $\mathbb{U}$: for every prefix-free machine M,

$$M(\sigma) = y \text{ implies } \mathbb{U}(\tau) = y,$$

for a string τ that is only by a constant d_M longer than σ.

Descriptive string complexity K

- The prefix-free Kolmogorov complexity of a string y is the length of a shortest $\mathbb{U}$-description of y:

$$K(y) = \min\{|\sigma| \colon \mathbb{U}(\sigma) = y\}.$$

- As a basic fact, $2^{-K(y)}$ is proportional to

$$\lambda\{X \in 2^{\mathbb{N}} \colon \mathbb{U}(\sigma) = y \text{ for some initial segment } \sigma \text{ of } X\},$$

where λ denotes product measure in Cantor space $2^{\mathbb{N}}$.

- Informally, this is the probability that $\mathbb{U}$ prints y.

K-trivial functions $f \colon \mathbb{N} \to \mathbb{N}$

Definition

A function $f \colon \mathbb{N} \to \mathbb{N}$ is called K-trivial if

$$\exists b \in \mathbb{N} \, \forall n \in \mathbb{N} \, [K(f \upharpoonright n) \leq K(n) + b].$$

Here $f \upharpoonright n$ denotes the tuple of the first n values of f, and we assume some effective encoding of tuples over $\mathbb{N}$ by natural numbers.

This extends the usual definition for sets (seen as $0, 1$-valued functions). Each computable function is K-trivial, but not conversely.

Proposition

A function f is K-trivial $\iff$ graph $\{\langle n, f(n) \rangle \colon n \in \mathbb{N}\}$ is K-trivial.

By the corresponding result for sets (N., 2003), each K-trivial function f is low: $f' \leq_T \emptyset'$.

K-trivial points

Let $\mathcal{M}$ be a computable metric space. Recall that a point $x \in M$ is called computable if it has a computable Cauchy name. This can be extended to any lowness property of functions. E.g. low points. Here is a different kind of generalisation.

Definition

A point $x \in M$ is called *K-trivial* if it has a K-trivial Cauchy name.

The following form computable metric spaces in a natural way:

- The unit interval.
- Baire space $\mathbb{N}^\mathbb{N}$ with the ultrametric distance function $d(f,g) = \max\{2^{-n} \colon f(n) \neq g(n)\}$.

In these spaces, a point is K-trivial iff it is K-trivial in the usual sense.

Preservation of K-triviality in metric spaces

A map F between computable metric spaces is called computable if there is a Turing functional Φ that turns every Cauchy name for x into a Cauchy name for $F(x)$.

Proposition

Let $\mathcal{M}, \mathcal{N}$ be computable metric spaces, and let the map $F\colon \mathcal{M} \to \mathcal{N}$ be computable. If x is K-trivial in $\mathcal{M}$, then $F(x)$ is K-trivial in $\mathcal{N}$.

- This relies on a hard result [N., 05], the downward closure under $\leq_T$ of the class of K-trivial sets.
- However, the result can be verified directly if F is Lipschitz.
- This already shows that K-triviality is invariant under the change of the computable structure to an equivalent one (i.e., when the identity map is computable in both directions).

A metric space where all K-trivial points are computable

Example

There is a computable metric space $\mathcal{M}$ such that

- $\mathcal{M}$ contains a noncomputable point.
- The only K-trivial points are the computable points.

Proof.

Ω denotes the measure of the domain of the universal prefix-free machine $\mathbb{U}$. Note that $\Omega \equiv_T \emptyset'$.

Ω_s denotes the measure of the domain of $\mathbb{U}$ at stage s. Let

$$M = \{\Omega_s \colon\ s \in \mathbb{N}\} \cup \{\Omega\}$$

with the metric inherited from the unit interval.

The computable structure is given by $\alpha_s = \Omega_s$.

If g is a Cauchy name for Ω then $\Omega \leq_{\mathrm{T}} g$, so g is not K-trivial. $\square$

Existence of K-trivials

Brattka and Gherardi, 2009:

If a computable metric space $\mathcal{M}$ has no isolated points, then there is a computable injective map $F\colon \{0,1\}^{\mathbb{N}} \to \mathcal{M}$ which is Lipschitz.

Theorem (Melnikov and N.)

Suppose a computable metric space has no isolated points.
Then it contains a K-trivial, non-computable point.

Proof.

- Let $A \in \{0,1\}^{\mathbb{N}}$ be a K-trivial non-computable set.
- Then $F(A)$ is K-trivial, where F is the function above.
- The inverse of F is computable on its domain. Hence the point $F(A)$ is non-computable.

□

The local condition

As usual we fix a computable metric space $\mathcal{M}$.

Letters p, q range over special points in $\mathcal{M}$.

We write $K(u, v)$ for $K(\langle u, v \rangle)$, the complexity of the ordered pair.

Definition

We say that $x \in M$ is locally K-trivial via b if

$$\forall n \, \exists p \text{ special } [d(x, p) \leq 2^{-n} \wedge K(p, n) \leq K(n) + b].$$

- Given a set $A \subseteq \mathbb{N}$, from $A \upharpoonright n$ we can determine n, but in general from a special point p we cannot determine the intended distance to x.
- So this definition is the appropriate analog of the usual definition $K(A \upharpoonright n) \leq K(n) + O(1)$ in Cantor space.
- The condition $K(p) \leq K(n) + b$ is insufficient by an example of Melnikov and N.

Equivalence with K-triviality

The local condition appears to be weaker, because we only require that pairs $\langle p, n\rangle$ be compressible, not the whole tuple of special points for distances down to 2^{-n}. However, this intuition is misleading:

Theorem

$x \in M$ is K-trivial $\Longleftrightarrow$ x satisfies the local condition for some constant b:

$$\forall n \, \exists p \text{ special } [d(x,p) \le 2^{-n} \wedge K(p,n) \le K(n) + b].$$

Proof idea, $\Longrightarrow$.

This is easy: let f be a K-trivial Cauchy name for x. Given n, let $p = f(n)$. Then

$$K(p,n) \le K(f \upharpoonright (n+1)) + O(1) \le K(n) + O(1).$$

Equivalence with K-triviality

Theorem (again)

$x \in M$ is K-trivial $\Longleftrightarrow$

$$\forall n \, \exists p \text{ special } [d(x,p) \leq 2^{-n} \wedge K(p,n) \leq K(n) + b].$$

Proof idea, $\Longleftarrow$. Recall that we identify special points with numbers.

- There are at most $O(2^b)$ many special p such that $K(p,n) \leq K(n) + b$.
- for sufficiently large N, there is a special point p_N as above that is the only such point at distance at most 2^{-N} from x.
- Fix a Solovay function h (i.e., h is computable, $K(n) \leq h(n)$, with equality infinitely often). Consider the infinite c.e. tree

$$T = \{(p_N, \ldots, p_r) \colon \, (p_i, p_{i+1}) \leq 2^{-i-1} \wedge K(p_i, i) \leq h(i) + b\}$$

for all i. An infinite path yields the required Cauchy name.

Dynamic characterization of K-trivial points

In recent work with Greenberg and Turetsky, we provide a dynamic characterization of K-trivial points via the amount of changes of a computable approximation.

Let $c_\Omega(x, s) = \Omega_s - \Omega_x$, where $x, s \in \mathbb{N}$. [N., 2014]: a set $A \subseteq \mathbb{N}$ is K-trivial iff it has a computable approximation with a finite total c_Ω-cost of changes. This cost is defined as $\sum_s c_\Omega(x_s, s)$, where least change is at x_s.

Metric version of c_Ω:

$$m_\Omega(\theta, s) = \Omega_s - \Omega_{\lfloor -\log \theta \rfloor},$$

for $\theta \in \mathbb{Q}^+$, $s \in \mathbb{N}$.

Theorem (With Greenberg and Turetsky)

A point $x \in M$ is K-trivial $\iff$ there is a computable sequence (p_s) of special points converging to x such that $\sum_s m_\Omega(d(p_s, p_{s+1}), s) < \infty$.

Left-c.e. metric spaces

The following is work with A. Gavruskin (LMJ, 2014).

A real β is called left-c.e. if the left cut $\{q \in \mathbb{Q} \colon\ q < \beta\}$ is c.e.

Definition

Let (M, d) be a complete metric space, and let $(\alpha_i)_{i \in \mathbb{N}}$ be a dense sequence in M.

- $\mathcal{M} = (M, d, (\alpha_i)_{i \in \mathbb{N}})$ is a left-c.e. metric space if $d(\alpha_i, \alpha_k)$ is a left-c.e. real uniformly in i, k.

Intuitively, the distance between special points can increase over time.

In this setting the Cauchy names form a Π^0_1 class in Baire space. This, along with the result to follow, suggests that being left-c.e. is more the natural notion for a distance function, rather than right c.e.

Examples

Let β be a left-c.e. real. Then $[0, \beta]$ and $\{z \in \mathbb{C} \colon |z| = \beta\}$ are naturally left-c.e. metric spaces.

Theorem (Gavruskin and N., LMJ, 2014)

Let $\gamma > 0$ be a left-c.e. real. Within the class of left-c.e. metric spaces of diameter at most γ, there is a left-c.e. metric space U which is universal with respect to computable isometric embeddings.

Co-c.e. equivalence relations can be seen as left-c.e. (pseudo)metric spaces where the distance between special points can only jump from 0 (equal) to 1 (different).

So our result extends a result of Ianovski, Miller, Ng and N. (JSL, 2014) that among the co-c.e. equivalence relations there is one that is universal with respect to computable embeddings.

The condition that the diameter be bounded is necessary:

Proposition (Gavruskin and N., LMJ, 2014)

Let $\mathcal{M} = (M, d, (\alpha_i)_{i \in \mathbb{N}})$ be a left-c.e. metric space. There exists a left-c.e. ultrametric space that cannot be isometrically embedded into M.

Future directions

- A main concept in the book by Pour-El/Richards is computable sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$. Study K-trivial sequences.
- The A.A. Markov/Ceitin (1950s) approach to computable analysis is based on computable points only. Relate this theory to K-triviality of points. E.g., if a function $F\colon \mathbb{R}_c \to \mathbb{R}_c$ is Markov computable, can it be continuously extended to the K-trivial points?
- Study K-trivial points in the more general context of left-c.e. metric spaces.